Linear Differential Equation with constant coefficients

Introduction :→

(i) The general form of a linear D.E. of n-th order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = Q \quad \text{—} (1)$$

Where $P_1, P_2, ----, P_n$ and Q are constants or functions of x (but not contain y).

Equations of First order :→

Let us take $n=1$ & $Q=0$. We then obtain

$$\frac{dy}{dx} + P_1 y = 0 \quad \text{—} (2)$$

$$\Rightarrow \frac{dy}{dx} = -P_1 y$$

$$\Rightarrow \frac{dy}{y} = -P_1 dx$$

Integrating, we have

$$-\log A + \log y = -P_1 x$$

$\therefore y = A e^{-P_1 x}$, Where A is an arbitrary constant.

Equations of second order :→

If $n=2$ & $Q=0$, equation ① becomes

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad \text{———} \quad (3)$$

The solution of equation ② suggest that $y = Ae^{mx}$, where m is constant, putting this value in equation ③

$$Ae^{mx}(m^2 + P_1 m + P_2) = 0$$

Thus, if m is a root of

$$m^2 + P_1 m + P_2 = 0 \quad \text{——} \quad (4)$$

$y = Ae^{mx}$ is a solution of equation ③, whatever may be the value of A

Let the equation ④ have two distinct real roots α & β. Then we have two solutions of equation ①, namely

$$y = Ae^{\alpha x} \text{ \& } y = Be^{\beta x}$$; Where A & B are arbitrary constants.

Now if substitude $y = Ae^{\alpha x} + Be^{\beta x}$ in the left side of the equation ③, we shall obtain

$$Ae^{\alpha x}(\alpha^2 + P_1\alpha + P_2) + Be^{\beta x}(\beta^2 + P_1\beta + P_2) = 0$$

Which is obviously 0 as α & β are the roots of equation ④.

Thus, $y = Ae^{\alpha x} + Be^{\beta x}$ is solution of equation (3) & since it contains two ~~arbitrary~~ independent arbitrary constants A & B, we may regard it as the general solution or complete primitive of the equation (3).

Example (1) :→ Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 54y = 0$

Solution :→ We take $y = Ae^{mx}$, a trial solution

$\therefore Ae^{mx}(m^2 + 3m - 54) = 0$

$\Rightarrow m^2 + 3m - 54 = 0$

$\Rightarrow m = 6, -9$

Therefor the general solution is

$$y = Ae^{6x} + Be^{-9x}$$

Where A & B are arbitrary constants.

Two special cases :→

(I) When the ~~axi~~ auxilary equation has imaginary roots.

When the auxiliary equation (4) has complex roots of the form $\alpha + i\beta$ & $\alpha - i\beta$ (α, β are real), then the general solution

$$y = Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x}$$

$$\Rightarrow y = e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[E\cos\beta x + F\sin\beta x\right]$$

Where $E = A + B$ & $F = i(A-B)$

Example (2) solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$

Solution:→

If the trail solution is taken as $y = Ae^{mx}$, the A.E. becomes

$$m^2 - 6m + 13 = 0$$

$$\Rightarrow m = 3 \pm 2i$$

∴ The general solution may be written as

$$y = Ae^{(3+2i)x} + Be^{(3-2i)x}$$

$$\Rightarrow y = e^{3x}(A\cos 2x + B\sin 2x)$$

Ans.

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(II) Case of equal roots :→

When the auxiliary equation has equal roots α and α, the solution becomes

$$y = Ae^{\alpha x} + Be^{\alpha x} = (A+B)e^{\alpha x}$$

Now $A+B$ is really only a single arbitrary constant.

Hence it cannot be regarded as the most general solution.

The general solution of this case will be

$$y = (A+Bx)e^{\alpha x}$$

A & B being arbitrary constants.

Example③ solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

Solution:→

The trail solution is $y = Ae^{mx}$

$\therefore A \cdot E \equiv m^2 + 4m + 4 = 0$

$\Rightarrow m = -2, -2$

$\therefore y = (A+Bx)e^{-2x}$

Ans.

Equations of order higher than the second

Let the equation be

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n = 0 \quad -(1)$$

And suppose the trial solution be $y = Ae^{mx}$.

The auxiliary equation is

$$m^n + P_1 m^{n-1} + \cdots\cdots + P_{n-1} m + P_n = 0 \quad -(2)$$

Rule I :→ Let $m_1, m_2, \cdots, m_n$ be n distinct real roots of (2). The general solution is then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots\cdots + c_n e^{m_n x}$$

Where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Rule-II :→ If two roots m_1 & m_2 of the auxiliary equation are equal, each equal to m, the corresponing part of the general solution is $(c_1 + c_2 x)e^{mx}$, & if three roots m_1, m_2, m_3 are each equal to m, the corresponding part of the solution is $(c_1 + c_2 x + c_3 x^2)e^{mx}$; & so on.

Rule-III :→

If the A.E. has a pair of imaginary roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, the corresponing part of the general solution is

$$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

or $c_3 e^{\alpha x} \cos(\beta x + c_4)$

or, $c_5 e^{\alpha x} \cdot \sin(\beta x + c_6)$

Any one of these form is used.

Rule IV:→ If a pair of imaginary roots $\alpha \pm i\beta$ occurs twice, the corresponding part of the general solution is

$$e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right]$$

Rule-IV :→

If the two real roots be m & $-m$, the corresponding parts of the solution is

$$c_1 e^{mx} + c_2 e^{-mx}$$

or, $c_1(\cosh mx + \sinh mx) + c_2(\cosh mx - \sinh mx)$

or, $c_3 \cosh mx + c_4 \sinh mx$

Example (4) Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

Solution :→

The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow (m+1)(m^2 - 4m + 4) = 0$$

$$\Rightarrow (m+1)(m-2)(m-2) = 0$$

$$\Rightarrow m = -1, 2, 2$$

∴ The general solution is

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-x}$$ Ans.

Example (5) solve $\frac{d^4y}{dx^2} + 8\frac{d^2y}{dx^2} + 16y = 0$

Solution :→

$$A \cdot E \equiv m^4 + 8m^2 + 16 = 0$$

$$\Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow m = \pm 2i, \pm 2i$$

∴ The general solution is

$$y = e^{0x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$

$$= (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Ans.

General Theory of Linear Differential Equations

The most general linear differential equation of n-th order is

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots + P_{n-1}\frac{dy}{dx} + P_n y = Q \quad \text{--- (1)}$$

Where the coefficients $P_1, P_2, \ldots, P_n$ and the right hand member Q are continuous function of x in $a \le x \le b$.

The fundamental ~~theorem~~ existance theorem proves that there exists a continuous solution $y(x)$ which assumes a given value y_0 at a point x_0 within $[a,b]$ and whose first $(n-1)$ derivatives are continuous & assume respectively the values

$$y_0', y_0'', \ldots\ldots, y_0^{(n-1)}$$

at $x = x_0$.

Introducing operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, etc, we write the equation (1) in the form

$$\{D^n + P_1D^{n-1} + \cdots\cdots + P_{n-1}D + P_n\}\,y = Q$$

will be called a linear differential operator of order n and will be denoted by $F(D)$.

Then equation (1) can be written as

$$F(D)y = Q \quad \text{--- (2)}$$

The equation

$$F(D)y = 0 \quad \text{--- (3)}$$

Where the right-hand member Q is supposed to be 0 is called the reduced equation (R.E.) of equation (1).

Properties of $F(D)$

(I) If $y = y_1$ is a solution of the reduced equation (3), then $y = cy_1$ is also a solution, where c is any arbitrary constant.

Proof:→ clearly $D^r(cy_1) = cD^r y_1$

Then

$$F(D)\,cy_1 = D^n(cy_1) + P_1 D^{n-1}(cy_1) + \cdots + P_{n-1}D(cy_1) + P_n \cdot cy_1$$

$$= c\{D^n y_1 + P_1 D^{n-1} y_1 + \cdots + P_{n-1}Dy + P_n y_1\}$$

$$= c\,F(D)\,y_1$$

∴ when $y = y_1$ is a solution of $F(D)y = 0$, the right-hand member vanishes and consequently

$$F(D)\,cy_1 = 0$$

i.e; cy_1 is also a solution of $F(D)y = 0$.

Case-II :- If $y = y_1, y_2, \cdots, y_m$ are m solutions of the reduced equation $F(D)y = 0$, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is a solution of the same equation ($c_1, c_2, \cdots, c_m$ are arbitrary constants)

Proof:→

$$\because D^r \{c_1 y_1 + c_2 y_2 + \cdots + c_m y_m\}$$

$$= c_1 D^r y_1 + c_2 D^r y_2 + \cdots + c_m D^r y_m$$

Now similarly argue ~~with~~ as in I

Case-III :→ Let $y = y_0(x)$ be any solution of equation ②, namely, $F(D)y = Q$. Then if $u(x)$ be the complete primitive of the reduced equation $F(D)y = 0$, $y = y_0(x) + u(x)$ will be the most general solution of $F(D)y = Q$.

Proof:→

Since the operator D^r is distributive, the linear operator $F(D)$ is distributive, that is to say

$$F(D)\{y_0(x) + u(x)\} = F(D)\{y_0(x)\} + F(D)\{u(x)\}$$

$$= Q + 0$$

$$= Q.$$

Moreover, the solution $y = y_0(x) + u(x)$ involves n arbitrary constants, it is therefor, the most general solution of $F(D)y = Q$.

Example ① Solve $\frac{d^2y}{dx^2} + 4y = 4x$

Solution :→ The equation

$$\frac{d^2y}{dx^2} + 4y = 4x$$

has the reduced equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

The complete primitive of which is

$y = A\cos 2x + B\sin 2x$.

Clearly $y = x$ is one solution of the given equation.

$\therefore \quad y = \text{C.F.} + \text{P.I.}$

$\quad = (A\cos 2x + B\sin 2x) + x$

Ans.